Problem set 1

Exercise 5:

(a) A curve with non-vanishing k is helix iff τ/k is Constant

$\Rightarrow$ α is helix, $\exists\, u$ (unit vector) s.t. $\langle T, u\rangle = $ Constant (*)
(call it $\cos\theta$)

$$0 = \langle T, u\rangle' = \langle T', u\rangle$$
$$= \langle k N, u\rangle$$
$$= k\langle N, u\rangle = 0$$

Since $k \neq 0$
$$\langle N, u\rangle = 0$$

$$0 = \langle N, u\rangle' = \langle N', u\rangle = \langle -kT + \tau B, u\rangle$$

$\Rightarrow$ If $u = \lambda(aT + bN + cB)$ $\quad \lambda \in \mathbb{R}\setminus\{0\}$
then $b = 0$
& $a = \tau$, $c = k$

$$u = \lambda(\tau T + kB)$$

from * & the fact u has unit norm

gives $u = \cos\theta\, T + \sin\theta\, B$

$$\Rightarrow \frac{d\tau}{dk} = \frac{\tau}{k} = \frac{\cos\theta}{\sin\theta} = \text{Constant}.$$

$\Leftarrow$ if $\frac{\tau}{k}$ is Constant

take $u = \cos\theta\, T + \sin\theta\, B$

where $\theta = \cot^{-1}\left(\frac{\tau}{k}\right)$ this is a fixed number by assumption

$$u' = \cos\theta\, T' + \sin\theta\, B'$$

$$= k\cos\theta\, N + \sin\theta\, {-\tau} N$$

$$= (k\cos\theta - \tau\sin\theta)\, N \quad = 0$$

because $\cos\theta = \frac{\tau}{\sqrt{k^2+\tau^2}}$ & $\sin\theta = \frac{k}{\sqrt{k^2+\tau^2}}$

(b) A curve is circle helix iff $\tau =$ Constant & $k =$ Constant > 0

If $\alpha = (r\cos t, r\sin t, ht)$ then simple Computations show $k = \frac{r}{r^2+h^2}$, $\tau = \frac{h}{h^2+r^2}$

The Converse follows from uniqueness part of fundamental theorem of space curves.

" Sorry I don't know the German name "

(c) α lies on a sphere if $\rho^2 + (\rho'\sigma)^2 = \text{Constant}$

$\rho = 1/k \quad , \sigma = \dfrac{1}{\tau}$

define $\quad m = \alpha + \rho N(s) + \rho'\sigma B(s)$

$$m' = T + \rho' N(s) + \rho N'(s) + \rho''\sigma B(s)$$
$$+ \rho'\sigma' B(s) + \rho'\sigma B'(s)$$

$$= T + \rho' N(s) + \rho(-\kappa T + \tau B) + \rho''\sigma B(s)$$
$$+ \rho'\sigma' B(s) + \rho'\sigma(-\tau N)$$

$$= (1 - \rho\kappa)T + (\rho' - \rho'\sigma\tau) N$$
$$+ (\rho\tau + \rho''\sigma + \rho'\sigma') B$$

Recall
$\rho = \dfrac{1}{\kappa}$

$\sigma = \dfrac{1}{\tau}$

$$= 0\,T + 0\,N + (\rho\tau + \rho''\sigma + \rho'\sigma') B$$

Enough to show $\quad \dfrac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma' = 0$

We have $\quad \rho^2 + (\rho'\sigma)^2 = \text{Costant}$, taking derivative

gives $\quad 2\rho\rho' + 2(\rho'\sigma)(\rho''\sigma + \rho'\sigma') = 0$

$$2\rho'\sigma\left(\frac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma'\right) = 0$$

$\sigma \neq 0$
$\rho' \neq 0$ $\quad \Rightarrow \left(\dfrac{\rho}{\sigma} + \rho''\sigma + \rho'\sigma'\right) = 0$

$\blacksquare$

Metric on a Surface

A metric on S is a "Smooth assignment" of inner product on each tangent space $T_p S$ for $p \in S$.

Eg: $\mathbb{R}^3$ naturally admits a metric
$T_p \mathbb{R}^3 \simeq \mathbb{R}^3$ for each $p \in \mathbb{R}^3$
and on this space we have standard inner product.

when $S \subseteq \mathbb{R}^3$ embedded, then there is a metric induced on S.

Given $p \in S \Rightarrow p \in \mathbb{R}^3$ and
$$T_p S \subseteq T_p \mathbb{R}^3$$

how to get $T_p S$ usually ?
locally around p, S can be given as a zero-set of some function $f : \underset{\text{in } \mathbb{R}^3}{\mathcal{U}} \longrightarrow \mathbb{R}$ with $p \in \mathcal{U}$.

Now, $T_p S$ is simply $\ker df_p$.

$$df_p = \text{Jac}(f) = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{pmatrix}$$

This inclusion of vector spaces

$T_pS \subseteq T_p\mathbb{R}^3$ induced a inner product on 2-dim vector space T_pS.

If x is a parametrization of S

ie, $x : \mathbb{R}^2 \longrightarrow \underset{\text{open}}{U} \subseteq S \subseteq \mathbb{R}^3$

$$(u,v) \longmapsto (x^1(u,v),\ x^2(u,v),\ x^3(u,v))$$

This is a homeomorphism ie, bijective continuous map with continuous inverse.

"S has a topology induced on it which is the subspace topology".

x induces a map $J_x : T_p\mathbb{R}^2 \longrightarrow T_{x(p)}S$

This map is the Jacobian map again, viewing x as a map into $\mathbb{R}^3$

$$x : \mathbb{R}^2 \longrightarrow U \subseteq S \longrightarrow \mathbb{R}^3$$

Thus $J_x\begin{pmatrix}1\\0\end{pmatrix} = \left(\dfrac{\partial x^1}{\partial u}(u,v),\ \dfrac{\partial x^2}{\partial u}(u,v),\ \dfrac{\partial x^3}{\partial u}(u,v)\right)$

and similarly

$$J_y\begin{pmatrix}0\\1\end{pmatrix} = \left(\frac{\partial x^1}{\partial v}(u,v) \, , \, \frac{\partial x^2}{\partial v}(u,v) \, , \, \frac{\partial x^3}{\partial v}(u,v) \right)$$

The elements $\overset{w_1}{\overset{\shortparallel}{J_x\begin{pmatrix}1\\0\end{pmatrix}}}$ & $\overset{w_2}{\overset{\shortparallel}{J_x\begin{pmatrix}0\\1\end{pmatrix}}}$ form a basis tangent space at every point in image of the map x.

In this basis the inner product on the tangent spaces (metric) is given by

$$g = \begin{pmatrix} \langle w_1, w_1 \rangle & \langle w_1, w_2 \rangle \\ \langle w_2, w_1 \rangle & \langle w_2, w_2 \rangle \end{pmatrix}$$

one we have g, the inner product is given by

$$\underset{\overset{\shortparallel}{v}}{(v_1 \quad v_2)} \begin{pmatrix} g \end{pmatrix}_{2\times 2} \underset{\overset{\shortparallel}{\tilde{v}}}{\begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix}}$$

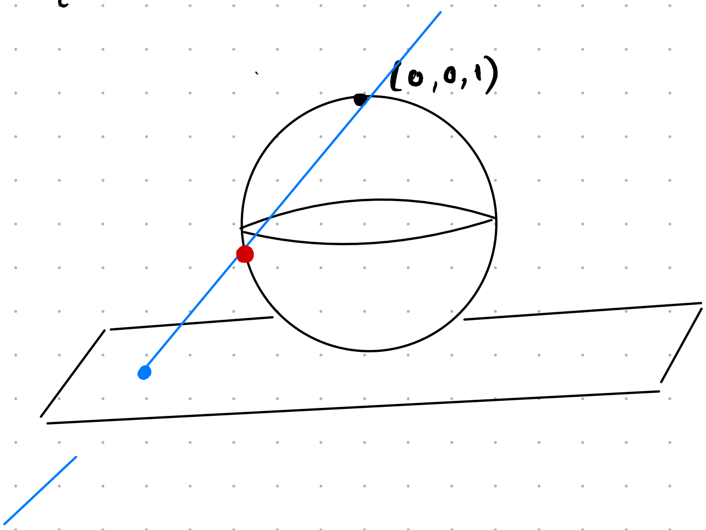
where v_1, v_2 are the Components of $v \in T_pS$ in the basis w_1 & w_2.

Problem Set 2

1. Stereographic projections.

$$S^2 = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2+y^2+z^2=1 \}$$

(a)



(b) $x : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is a parametrization

$$x(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$$

$t(u, v, 0) + (1-t)(0,0,1)$ line joining $(u,v,0)$ & $(0,0,1)$

$$\shortparallel$$

$$(tu, tv, (1-t))$$

find t such that $t^2u^2 + t^2v^2 + (1-t)^2 = 1$

$$u^2+v^2 + \left(\frac{1}{t}-1\right)^2 = \frac{1}{t^2}$$

$$u^2+v^2+ \frac{1}{t^2} - \frac{2}{t} + 1 = \frac{1}{t^2}$$

$$u^2+v^2 - \frac{2}{t} + 1 = 0$$

$$u^2+v^2+1 = \frac{2}{t}$$

$$t = \frac{u^2+v^2+1}{2}$$

$\Rightarrow \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$ is the unique point on S^2 intersecting this line (obviously other than $(0,0,1)$)

(c) $y : \mathbb{R} \longrightarrow \mathbb{R}^3$

$$y(\bar{u}, \bar{v}) = \left(\frac{2\bar{u}}{\bar{u}^2+\bar{v}^2+1}, \frac{2\bar{v}}{\bar{u}^2+\bar{v}^2+1}, \frac{1-\bar{u}^2+\bar{v}^2}{\bar{u}^2+\bar{v}^2+1} \right)$$

what is $x^{-1} \circ y : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

for this we will show $x^{-1}(a,b,c) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$

Verify its the x^{-1}!

now compute $x^{-1} \circ y$ & show its smooth

$$\boxed{x^{-1} \circ y\,(u,v) = \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right)}$$

(d) Thus by definition of surfaces, S^2 is a smooth surface.

(e)

$$x(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1} \right)$$

$$x_u(u,v) = \left(\frac{2v^2-2u^2+2}{(u^2+v^2+1)^2}, \frac{-4uv}{(u^2+v^2+1)^2}, \frac{4u}{(u^2+v^2+1)^2} \right)$$

$$x_v(u,v) = \left(\frac{-4uv}{(u^2+v^2+1)^2}, \frac{2u^2-2v^2+2}{(u^2+v^2+1)^2}, \frac{4v}{(u^2+v^2+1)^2} \right)$$

$$g_{21}(u,v) = g_{12}(u,v) = 0$$

$$g_{11} = \frac{4}{(u^2+v^2+1)^2}$$

$$g = \begin{pmatrix} \frac{4}{(u^2+v^2+1)^2} & 0 \\ 0 & \frac{4}{(u^2+v^2+1)^2} \end{pmatrix}$$

$$g_{22} = \frac{4}{(u^2+v^2+1)^2}$$

2. $\alpha(t) = (r(t), z(t))$ $t \in (a,b)$ $r(t) > 0$

This is rotated about z-axis, this gives us a surface called rotation surface S.

now we have a parametrization for S

$$x(t,\varphi) = \left(r(t)\cos\varphi,\ r(t)\sin\varphi,\ z(t) \right)$$

t curves are called meridians.
φ curves are called latitudes.

(a) if α is regular and injective, then x is a parametrization.

Injectivity of α clearly gives injectivity of the parametrization $x : \mathbb{R}^2 \to \mathbb{R}^3$

$$x_t(t,\varphi) = (\dot r(t)\cos\varphi,\ \dot r(t)\sin\varphi,\ \dot z(t))$$

$$x_\varphi(t,\varphi) = (-r(t)\sin\varphi,\ r(t)\cos\varphi,\ 0)$$

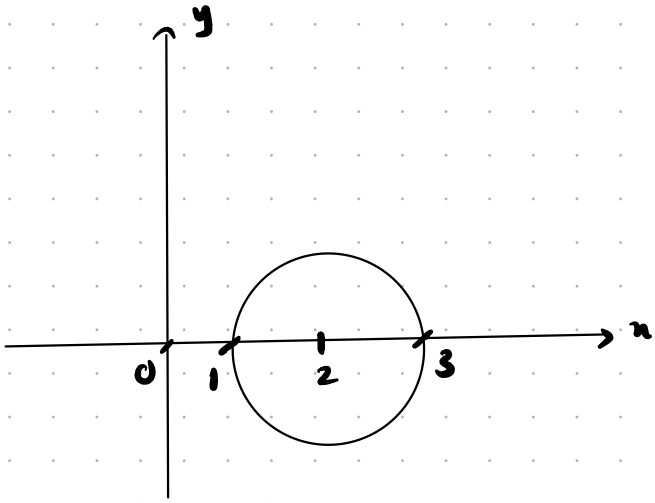
$$n(t,\varphi) = x_t(t,\varphi) \times x_\varphi(t,\varphi)$$
$$= (r(t)\dot z(t)\cos\varphi,\ -r(t)\dot z(t)\sin\varphi,\ \dot r(t)\, r(t))$$

$$|n(t,\varphi)|^2 = r^2(t)\left(\dot z^2(t) + \dot r^2(t)\right) > 0$$

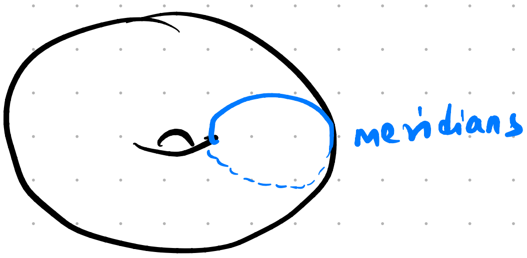
This means x_t & x_t are never parallel
if α is regular

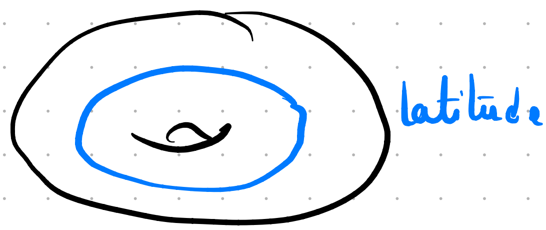
(b) $\alpha(t) = (r(t), z(t)) = (2+\cos t, \sin t)$ $t \in (-\pi, \pi)$

what is the rotation surface S.



when this is rotated we get a Torus





(c) metric in rotation surfaces

$$x(t,\varphi) = \left(r(t)\cos\varphi,\ r(t)\sin\varphi,\ z(t)\right)$$

$$x_t(t,\varphi) = \left(\dot r(t)\cos\varphi,\ \dot r(t)\sin\varphi,\ \dot z(t)\right)$$

$$x_\varphi(t,\varphi) = \left(-r(t)\sin\varphi,\ r(t)\cos\varphi,\ 0\right)$$

$$g_{11} = \dot r(t)^2 + \dot z(t)^2$$

$$g_{12} = 0$$

$$g_{21} = 0$$

$$g_{22} = r(t)^2$$

$$g = \begin{pmatrix} \dot r(t)^2 + \dot z(t)^2 & 0 \\ 0 & r(t)^2 \end{pmatrix}$$

3. $x(r,\varphi) = (r\cos\varphi,\ r\sin\varphi,\ r)$

$r \in \mathbb{R}^+$

$\varphi \in (0, 2\pi)$

from previous problem we have 'r' instead of t.

So $g = \begin{pmatrix} 2 & 0 \\ 0 & r^2 \end{pmatrix}$ (notice there is a problem at $r=0$)

$r(t) = e^{\frac{t\cos\theta}{2}}$, $\varphi(t) = \frac{t}{\sqrt{2}}$ this gives a curve in (S, g)

for a fixed θ & $t \in [0, \pi]$

what is length ?

$$L(\alpha) = \int_0^{\pi} |\dot{\alpha}(t)| \, dt$$

$$= \int_0^{\pi} \sqrt{g_{ij} \dot{\alpha}^i \dot{\alpha}^j} \, dt$$

$$= \int_0^{\pi} \sqrt{2(\dot{\alpha}^1)^2 + r^2 (\dot{\alpha}^2)^2} \, dt$$

$$= \int_0^{\pi} \sqrt{2\left(\frac{\cot\theta}{2} \exp\left(\frac{t\cot\theta}{2}\right)\right)^2 + \exp(t\cot\theta) \cdot \frac{1}{2}} \, dt$$

$$= \sqrt{\frac{\cot^2\theta + 1}{2}} \int_0^{\pi} \sqrt{\exp(t\cot\theta)} \, dt$$

$$= \frac{1}{\sqrt{2}\sin\theta} \frac{2}{\cot\theta} \exp \frac{t\cot\theta}{2} \Big|_0^{\pi}$$

$$= \frac{\sqrt{2}}{\sin\theta \cot\theta} \left(\exp \frac{\pi\cot\theta}{2} - 1 \right)$$

what is the angle α and φ = constant curve.

what is φ = constant curve

a vector in that direction would be

$w = \left(\cos\frac{t}{\sqrt{2}}, \sin\frac{t}{\sqrt{2}}, 1 \right)$ at time $\frac{t}{\sqrt{2}}$

claim is $\dfrac{\langle \alpha', w\rangle}{|\alpha'|\,|w|} = \cos\theta$ for all t & ϕ.

$$\alpha' = \Big(e^{\frac{t\cot\theta}{2}} \cdot \frac{\cot\theta}{2} \cos\frac{t}{\sqrt2} + -e^{\frac{t\cot\theta}{2}} \frac{1}{\sqrt2} \sin\frac{t}{\sqrt2},$$

$$e^{\frac{t\cot\theta}{2}} \frac{\cot\theta}{2} \sin\frac{t}{\sqrt2} + e^{\frac{t\cot\theta}{2}} \frac{1}{\sqrt2} \cos\frac{t}{\sqrt2},$$

$$e^{\frac{t\cot\theta}{2}} \frac{\cot\theta}{2} \Big)$$

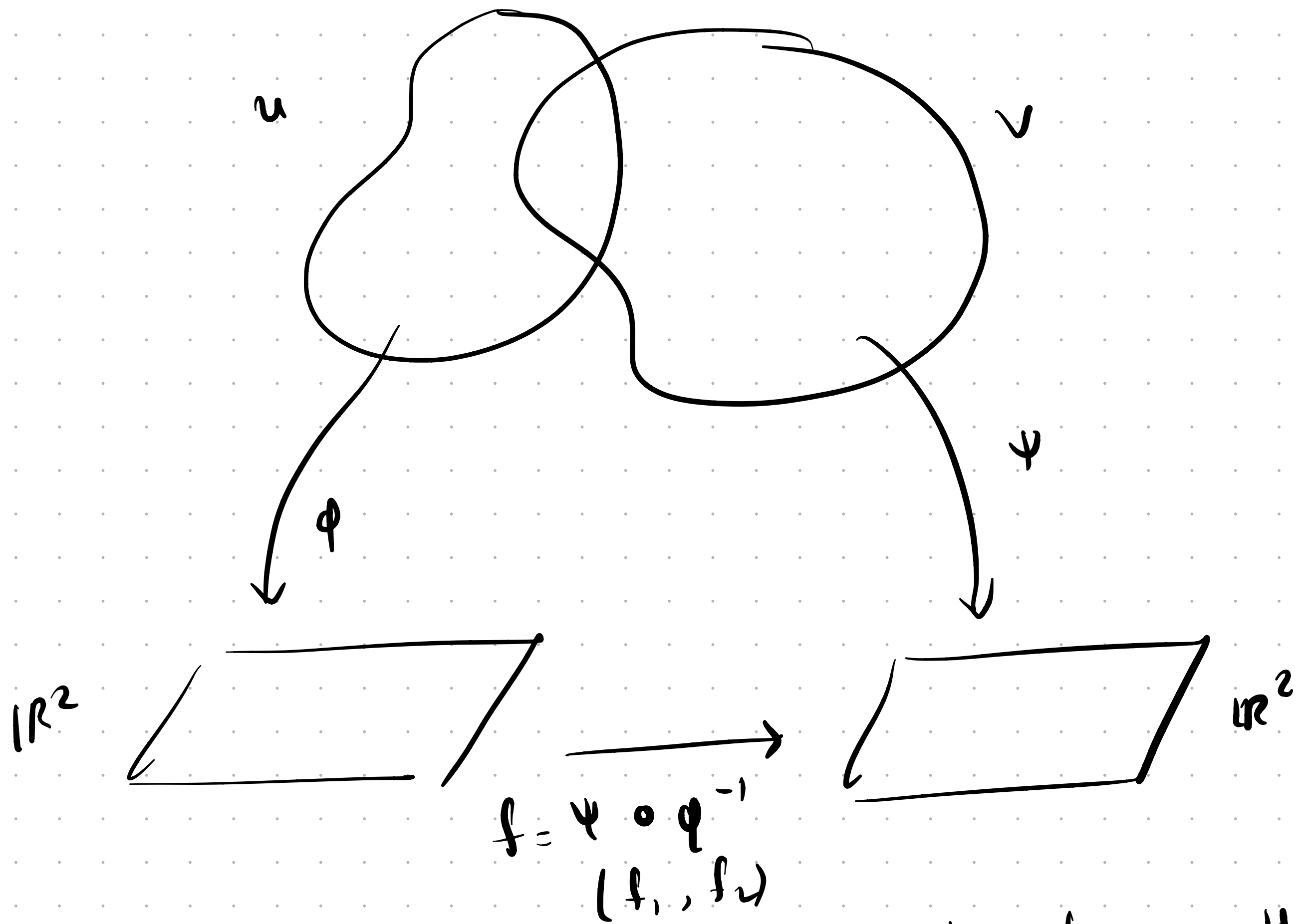
$$\frac{\langle \alpha', w\rangle}{|\alpha'|\,|w|} = \frac{2 e^{\frac{t\cot\theta}{2}} \frac{\cot\theta}{2}}{\sqrt{2\left(\frac{\cot\theta}{2} \exp\left(\frac{t\cot\theta}{2}\right)\right)^2 + \exp(t\cot\theta) \cdot \frac{1}{2}}\ \sqrt2}$$

$$= \frac{\cot\theta \exp\left(\frac{t\cot\theta}{2}\right)}{\sqrt{(\cot^2\theta + 1)\exp(t\cot\theta)}}$$

$$= \sin\theta\cot\theta = \cos\theta.$$

$\blacksquare$

(4) Transformation law for tangent vectors and the metric Coefficients.



The Jacobian matrix of f tells how the basis changes.

$$F = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

If the basis change is given by this matrix how does the g change.

$$\sum_{i,j} g_{ij}^{v} \alpha_i^{v} \beta_j^{v} = \sum_{i,j} g_{ij}^{v} (F\alpha^{u})_i (F\beta^{u})_j$$

$$= [F\alpha^u]^T \begin{bmatrix} & & \\ & g & \\ & & \end{bmatrix}^v [F\beta^u]$$

$$= \alpha^u \underline{F^T G^v F} \beta^u$$

Thus G^v changes as $F^T G^v F$

$$G^u = F^T G^v F$$

& $\alpha^v = F\alpha^u$ $\blacksquare$